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MINIMUM-WEIGHT DESIGN OF MULTI-PURPOSE CYLINDRICAL BARS

N. V. BANICHUK
Institute of Problems of Mechanics, USSR Academy of Sciences, Moscow, USSR

and

B. L. KARIHALOO
Department of Solid Mechanics, The Technical University of Denmark, 2800 Lyngby, Denmark

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Abstract—We solve the problem of minimizing the weight of elastic cylindrical bars under given constraints on their torsional and bending rigidities. The possible types of solutions are classed in the space of the design parameters. The necessary optimality conditions are derived and then used in the formulation of a closed-form boundary-value problem for a region with an unspecified boundary—the unknown cross-sectional shape of the optimal bar. An analytical solution of this and other related problems in terms of the given design parameters is obtained.

INTRODUCTION

Optimal design problems usually consist in determining the shape of a minimum-weight structure whose mechanical properties fulfil certain specified requirements such as restrictions on its stressed (strained) state, dynamic characteristics (such as the frequencies of natural vibrations), buckling load, etc. The great majority of the problems attempted so far involved structures that were required to perform some single function (so-called single-purpose structures, see, for example, the review papers [1, 2]). Thus, for example, it was required to determine the shape of a minimum-weight elastic structure under the condition that the fundamental frequency of its natural vibrations exceeded a certain given quantity. Likewise, dual optimal design problems were considered, whereby some mechanical property was optimized (maximized or minimized) for a given weight of the structure (so-called quality criteria). Like the direct problems, the dual ones were also usually reduced to the solution of an isoperimetric variational problem with a single constraint.

However, in the optimal design process, as in the conventional one, it is often necessary to include the actual working conditions of the designed structure (this involves a corresponding increase in the number of parameters) and to design rather versatile structures that fulfil various requirements (multi-purpose structures). Some simple examples of multi-purpose optimal design are given in [3, 4]. It may be mentioned that multi-purpose design problems arise not only in deterministic situations, where it is desired to design a structure to given specifications (such as constraints on strength, geometry and frequencies) and for various loading conditions, but also in situations, in which a complete information about external influences (loadings) is absent necessitating consideration of all possible practical realizations of such influences [5].

The present paper deals with the problem of minimizing the cross-sectional area (the weight) of elastic cylindrical bars under given constraints on their torsional and bending rigidities. Such a multi-purpose optimization problem differs from those studied earlier [3–6] in that it is no longer a one-dimensional problem and is described by partial differential equations, the partial derivatives entering both the differential equation and the necessary optimality condition. A closed-form solution of this and related problems is given and some important implications of the method are discussed.

1. STATEMENT OF THE PROBLEM

The following optimal design problem is posed. It is required to minimize the cross-sectional area S (the weight) of an elastic cylindrical bar whose torsional (K) and bending (C) rigidities satisfy the constraints $K \geq K_0$, $C \geq C_0$, where K_0 and C_0 are given constants. It should, however, be stressed that the bar is not expected to withstand external twisting and bending moments simultaneously, but that during its design life it may be called upon to transmit twisting moments part of the time, while at other times it may be expected to act like a beam. Such situations are not as rare in practice as may at first appear to be the case. In particular, they may arise at the design stage when insufficient information is available on the actual working conditions to be faced by the structure and allowance has to be made for various possibilities. A similar situation may exist if the actual loading programme of the structure (for example, a programme involving twisting followed by bending and so on) is specified beforehand.

We begin with a rigorous mathematical formulation of the problem. In this connection we consider initially one by one the various design purposes of the structure. Thus, if the cylindrical bar is required to transmit twisting moments M applied at its ends (i.e. it is working in pure shear conditions), the torsional rigidity K, the angle of twist per unit length of the bar θ and M must be related through $M = K\theta$. To formulate this problem mathematically, we introduce a stress function $u = u(x, y)$ which is a solution of the boundary-value problem[7]

$$\begin{aligned} u_{xx} + u_{yy} &= -2 \quad x, y \in D \\ u &= 0 \quad\;\; x, y \in \Gamma, \end{aligned} \tag{1.1}$$

where D is the simply-connected region of integration (cross-section of the bar normal to its length) and Γ is the boundary of the domain D, while subscripts denote differentiation with respect to corresponding spatial variables.

The torsional rigidity K is expressed through $u(x, y)$

$$K = 2G\int_D \int u \, dx \, dy, \tag{1.2}$$

G being the shear modulus. The design constraint has the form

$$K \geq K_0. \tag{1.3}$$

Next, we consider the case in which the bar acts like a beam. The primary mechanical property of a beam is its bending stiffness C. Assuming the bending to take place in the plane yz (z axis is directed along the length of the beam), C may be written as

$$C = E\int_D \int y^2 \, dx \, dy, \tag{1.4}$$

E being the elastic modulus.

It is well-known that the bending stiffness (of a cylindrical bar) characterizes not only its statical deflection pattern under transverse loads, but also the frequency spectrum of its free vibrations. Thus, the greater the bending stiffness of the beam, the less will it deflect under statical loads and the higher will be the fundamental frequency of its natural vibrations (for a given mass). Therefore, the constraints on the maximum allowable deflection and on the dynamic characteristics can be reduced to the following design requirement

$$C \geq C_0. \tag{1.5}$$

The optimization problem at hand consists in seeking the cross-sectional shape of the elastic bar that fulfils the requirements (1.3) and (1.5) and has the least area S

$$S(\Gamma) = \int_D \int dx \, dy \to \min. \tag{1.6}$$

The mathematical problem is thus reduced to one with integral (1.2)–(1.5) and differential (1.1) constraints.

2. TYPES OF SOLUTIONS AND NECESSARY OPTIMALITY CONDITION

Let us analyse the types of solutions possible for the problem formulated. To do this, we consider first the problem (1.1)–(1.3), (1.6), i.e. the problem of minimizing the cross-sectional area of a bar subject to only a single constraint—that on its torsional rigidity. It is not difficult to show that this problem has a dual, viz, the problem of maximizing the torsional rigidity for a given area S. The optimal solution of the problem (like that of the dual one, [8]) is therefore a bar of circular cross-section with radius $r = (2K_0/\pi G)^{1/4}$.

In general, the torsional and bending rigidities of a circular bar are related through $K = 2GC/E = C/(1+\nu)$, ν being Poisson ratio. This expression corresponds to a straight line in the plane $K \geq 0$, $C \geq 0$ (Fig. 1), the slope of the line varying from 45° to 33·7° for $0 \leq \nu \leq 1/2$. Every bar of circular cross-section corresponds to this line in the plane (K, C). If the specified design constraints K_0, C_0 satisfy the inequality

$$K_0 \geq 2\frac{G}{E}C_0, \tag{2.1}$$

i.e. they lie in the region I (Fig. 1), the optimal solution is a circular bar. This follows from the fact that if the inequality (2.1) is observed by a circular bar of torsional rigidity K_0, the bending rigidity of such a bar must automatically exceed C_0, i.e. the condition (1.5) is oversatisfied.

On the other hand, however, if the given design parameters violate the inequality (2.1), i.e. they lie in the region II, both the constraints are of importance in deciding the shape (assumed convex) of the optimal bar. They can be included in the mathematical formulation through the use of the Lagrange multiplier technique.

It should be noted, that the plane $K \geq 0$, $C \geq 0$ contains no other regions in which the optimal design is controlled only by inequality (1.5), inasmuch as, without violating the convexity condition, it is possible to reduce the cross-sectional area of the bar indefinitely while keeping its bending stiffness constant. This is easily explained by the example of a rectangular bar of width b and depth h, for which $S = bh$ and $C = Ebh^3/12$. Assuming the bending stiffness to be given and equal to C_0 and eliminating b from the expressions for S and C, we get $S = 12C_0/Eh^2$, whence it follows that as h is increased, the area of cross-section can be reduced indefinitely.

It is thus necessary to seek a solution to the optimal design problem only in the region II. Before proceeding to derive the necessary optimality condition, it is expedient to reformulate the problem (1.1)–(1.6) by eliminating the differential constraint (1.1). This is easily accomplished through the use of well-known results from the theory of elastic bars under torsion [7] and from [9]. Without going into detail, it may be shown that

$$K = \min_u \frac{G}{2}\int_D \int (u_x^2 + u_y^2 - 4u)\,dx\,dy \geq K_0, \tag{2.2}$$

where the minimum with respect to u is sought among a class of continuously differentiable functions that satisfy the boundary condition $u = 0$ on Γ. The functional (2.2) includes the differential equation (1.1), being its Euler equation, and thus brings this into picture only implicitly.

To derive the necessary optimality condition for the problem (1.4)–(1.6), (2.2), we employ the Lagrange multiplier technique and construct an auxiliary functional

$$\Pi = \int_D \int dx\,dy + \lambda_1 \frac{G}{2}\int_D \int (u_x^2 + u_y^2 - 4u)\,dx\,dy + \lambda_2 E \int_D \int y^2\,dx\,dy \tag{2.3}$$

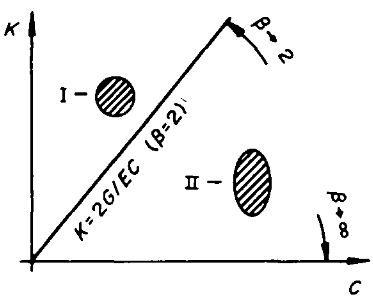

Fig. 1.

(the terms $\lambda_1 K_0$ and $\lambda_2 C_0$ are omitted), and write the expression for its first variation

$$\delta\Pi = \int_\Gamma \mathrm{d}f\,\mathrm{d}s + \lambda_1\frac{G}{2}\int_\Gamma (u_x^2 + u_y^2 - 4u)\,\mathrm{d}f\,\mathrm{d}s + \lambda_2 E\int_\Gamma y^2\,\mathrm{d}f\,\mathrm{d}s + \lambda_1 G\int_D\int \delta u(u_{xx} + u_{yy} + 2)\,\mathrm{d}x\,\mathrm{d}y, \quad (2.4)$$

where $\mathrm{d}f$ denotes the normal displacement of points on the boundary due to a variation of the region D. As $u = 0$ on Γ and $\mathrm{d}f$ is arbitrary, the stationarity property $\delta\Pi = 0$ gives the necessary optimality condition

$$1 + \lambda_1\frac{G}{2}(u_x^2 + u_y^2) + \lambda_2 E y^2 = 0,$$

which may be rewritten in a more convenient form

$$u_x^2 + u_y^2 = \mu_1 + \mu_2 y^2, \quad (2.5)$$

with $\mu_1 = -2/\lambda_1 G$ and $\mu_2 = -2\lambda_2 E/\lambda_1 G$. These constants are determined using (1.3) and (1.5) with $K = K_0$ and $C = C_0$. Thus, the solution of the optimization problem (the cross-sectional shape of least area and the corresponding stress function $u(x, y)$) can be found from the complete system of equations (1.1)–(1.5) and (2.5), the equality sign holding in (1.3) and (1.5).

3. ANALYTICAL SOLUTION OF THE PROBLEM

Guided by the necessary optimality condition (2.5), let us assume a solution of the type

$$\Gamma: x^2 + cxy + ay^2 = b$$
$$u = (b - x^2 - cxy - ay^2)N,$$

where the real constants a, b, c, N are to be determined together with the Lagrange multipliers μ_1 and μ_2. The assumed form of Γ and u is such that the boundary condition $u = 0$ on Γ is identically satisfied. Evidently, if there exists a solution of the type (3.1), the constants must be uniquely defined by (1.1)–(1.5) and (2.5). The optimality condition (2.5) would require the cross-section of the least area to be of the form

$$(4 + c^2)x^2 + (c^2 + 4a^2 - \mu_2/N^2)y^2 + 4c(1 + a)xy = \mu_1/N^2.$$

Equivalence between the assumed shape of the contour Γ and that specified by the optimality condition would demand that

$$4 + c^2 = 1; \quad (c^2 + 4a^2 - \mu_2/N^2) = a$$
$$4(1 + a) = 1; \quad \mu_1/N^2 = b,$$

whence it follows that a and c are respectively negative real and imaginary quantities—in violation of the assumptions of convexity and simply-connectedness of Γ. Moreover, by dividing the above equation throughout by $(4 + c^2)$, it is easy to show that for c to be a non-zero real quantity, either $\mu_2 = 0$ or $N = \infty$, both of which are unacceptable on physical grounds. Thus it is evident that the product term must be absent in the assumed form of Γ.

From the above considerations it follows that the solution should be of the form

$$\Gamma: x^2 + ay^2 = b$$
$$u = (b - x^2 - ay^2)N. \quad (3.1)$$

As the optimal cross-section transpires to be elliptical in shape the constants a and b could be determined directly from simple expressions for the bending and torsional rigidities of an elliptical section. However, in order to specify the stress function uniquely we determine a, b, N, μ_1 and μ_2 from the following complete system of simultaneous equations

$$1 + a = 1/N \tag{3.2}$$

$$a^2 - a - \mu_2/4N^2 = 0; \quad \mu_1/4N^2 = b, \tag{3.3}$$

$$b^2N/a^{1/2} = K_0/G\pi, \tag{3.4}$$

$$b^2/a^{3/2} = 4C_0/E\pi, \tag{3.5}$$

where (3.2) follows from the differential eqn (1.1), (3.3)—from a comparison of the optimality condition (2.5) with the assumed form of Γ (3.1), and (3.4) and (3.5) from the constraints (1.2) and (1.4) (with $K = K_0$ and $C = C_0$). In obtaining (3.4) and (3.5) it is convenient to map the region of integration D bounded by Γ on to a unit circle through a linear transformation and to change to polar coordinates.

Solving the system of simultaneous eqns (3.2)–(3.5) in a, b, N, μ_1 and μ_2, we get

$$\begin{aligned} &a = 1/(\beta - 1), \quad b = (K_0\beta/\pi G)^{1/2}(1/\beta - 1)^{3/4},\\ &N = (\beta - 1)/\beta, \quad \mu_1 = 4(K_0/\pi G)^{1/2}(\beta - 1)^{5/4}/\beta^{3/2},\\ &\mu_2 = 4(2 - \beta)/\beta^2, \end{aligned} \tag{3.6}$$

where $\beta = 4C_0G/EK_0$. Note that, in view of the inequality (2.1), $\beta \geq 2$ in the region II. The shape of the cross-section of least area and the corresponding stress function $u(x, y)$ take the form

$$\begin{aligned} &\Gamma: x^2 + (1/\beta - 1)y^2 = (K_0\beta/\pi G)^{1/2}(1/\beta - 1)^{3/4},\\ &u = [(K_0\beta/\pi G)^{1/2}(1/\beta - 1)^{3/4} - x^2 - (1/\beta - 1)y^2](\beta - 1)/\beta. \end{aligned} \tag{3.7}$$

Given the material constants E, G and the design constraints K_0, C_0 (satisfying the requirement $4C_0G/EK_0 \geq 2$), the shape of the cylindrical bar that has the least area (weight) of cross-section can easily be established using (3.7), the area S_{opt} being

$$S_{opt} = (\pi K_0/G)^{1/2}(\beta^2/\beta - 1)^{1/4}. \tag{3.8}$$

In order to assess the economy achieved by optimizing the cross-sectional shape, let us compare S_{opt} with that of a circular bar of the same bending stiffness C_0 as the optimal bar. It may be remarked that, in region II, the torsional rigidity of a circular bar with bending stiffness C_0 will always exceed K_0. The cross-sectional area S_0 of such a bar in terms of C_0 is

$$S_0 = (4\pi C_0/E)^{1/2}, \tag{3.9}$$

whence, by using (3.8), it follows that

$$\frac{S_0 - S_{opt}}{S_0} = 1 - 1/(\beta - 1)^{1/4}. \tag{3.10}$$

It is evident from (3.10) that for $\beta = 2$ (at the limit of validity to (3.7)), S_0 is equal to S_{opt}, i.e. a circular bar is the optimal one, while for $\beta > 2$ it is possible to achieve a substantial economy in the cross-sectional area (weight) of the bar.

4. SOLUTION OF RELATED PROBLEMS

There are two other problems which can be traced to the optimality condition derived above (2.5). One of these involves the maximization of the torsional rigidity ($K \to \max$) of an elastic cylindrical bar subject to given constraints on the area (weight) of its cross-section and the bending stiffness

$$C \geq C_0, \quad S \leq S_0. \tag{4.1}$$

The other problem deals with the maximization of the bending stiffness ($C \to \max$) under given constraints on the area of cross-section and the torsional rigidity

$$K \geq K_0,$$
$$S \leq S_0. \tag{4.2}$$

For the first problem the plane ($C \geq 0$, $S \geq 0$) of the design parameters (Fig. 2) is divided into two regions by the parabola $C = ES^2/4\pi$. Moreover, if the specified design parameters C_0, S_0 satisfy the inequality

$$C_0 \leq ES_0^2/4\pi, \tag{4.3}$$

i.e. they lie in the region I (Fig. 2), the cross-section of the bar of maximum torsional rigidity will be circular in shape with radius $r = \sqrt{\pi S_0}$. If, however, the constraints violate the inequality (4.3) and thus fall in the region II, it is clear that both constraints will play their part in describing the cross-sectional shape. As before, the optimal cross-section is elliptical in shape. Performing calculations similar to those in Section 3, we arrive at

$$\Gamma: x^2 + \alpha y^2 = S_0(\alpha)^{1/2}/\pi$$
$$u = (S_0(\alpha)^{1/2}/\pi - x^2 - \alpha y^2)(1/1+\alpha), \tag{4.4}$$

where $\alpha = (ES_0^2/4C_0\pi)^2$, the corresponding torsional rigidity K_{opt} being

$$K_{opt} = GS_0^2(\alpha)^{1/2}/\pi(1+\alpha). \tag{4.5}$$

Consider now the second problem—that of maximizing the bending stiffness of a cylindrical bar subject to the conditions (4.2). As in the previous problem, the plane $K \geq 0$, $S \geq 0$ of the design parameters (Fig. 3) is divided into two regions by the parabola $K = GS^2/2\pi$. However, in contradistinction to the earlier two problems, if the specified constraints K_0, S_0 are such that

$$K_0 > GS_0^2/2\pi, \tag{4.6}$$

i.e. they lie in the hatched region (Fig. 3), the problem at hand does not have a solution. This is explained by the fact that, for the given relationship (4.6) between K_0 and S_0 to hold, the torsional rigidity of a bar of no matter what cross-sectional shape (including circular) must be less than the given value K_0. In the remaining part of the plane, in which $K_0 \leq GS_0^2/2\pi$, the optimal elliptical cross-section and the corresponding stress function take the form

$$\Gamma: x^2 + ay^2 = S_0(a)^{1/2}/\pi$$
$$u = (S_0(a)^{1/2}/\pi - x^2 - ay^2)(1/1+a), \tag{4.7}$$

the corresponding bending stiffness C_{opt} being

$$C_{opt} = S_0^2E/4\pi a^{1/2}, \tag{4.8}$$

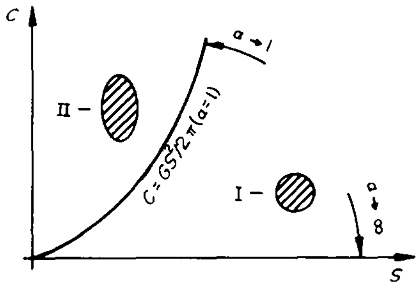

Fig. 2.

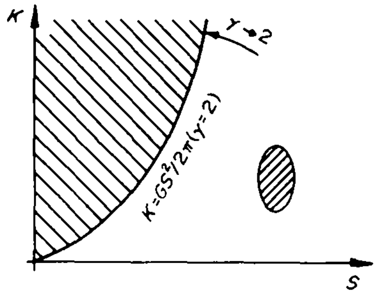

Fig. 3.

where

$$a^{1/2} = \gamma/2 - (\gamma^2/4 - 1)^{1/2}, \tag{4.9}$$

and

$$\gamma = GS_0^2/\pi K_0.$$

The negative sign before the radical in (4.9) has been chosen with a view to securing the maximum bending stiffness in the plane yz, (i.e. the longer of the two semi-axes of the elliptical cross-section is directed along the y-axis). A positive sign would correspond to the lowest bending stiffness for a bar of given cross-sectional area S_0 and torsional rigidity K_0.

In conclusion, it may be remarked that the present method, involving an analysis of the possible types of solutions, can be generalized to other multi-purpose optimal design situations and to problems with a larger number of constraints. Moreover, within the framework of the present mathematical formulation, it is possible to cover the optimization problem for a bar with multiply-connected (hollow) cross-section (under the assumption that the shape of some of the contours is prescribed, while that of the rest is to be determined). In this case the necessary optimality condition (2.5) remains unaltered, but an effective method of solution would be a perturbation technique similar to that used in [9].

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